MATHEMATICS OF COMPUTATION, VOLUME 29, NUMBER 132
OCTOBER 1975, PAGES 1129–1134

The Distribution of Ideal Class Numbers of Real Quadratic Fields

By M. D. Hendy

Abstract. A table of class numbers of real quadratic number fields $Q(\sqrt{d})$ with square-free determinant d, $1000 < d < 100000$ is examined and several analyses of the distribution of the class numbers, and the number of classes per genus are made. From these, two conjectures on the possible distribution of the class numbers as $d \to \infty$ are made, which are consistent with Gauss's related conjecture.

1. Introduction. In 1965 K. E. Kloss [4] announced the production of a table listing the primes $p \equiv 1 \pmod 4$, $p < 120000$ which are the determinants of real quadratic number fields whose domains of integers are unique factorization domains. He notes that approximately 80% of the primes considered are in the table. In addition, a table of ideal class numbers $h(p)$ for $Q(\sqrt{p})$, $p \equiv 1 \pmod 4$, p prime < 95000 was produced.

Subsequently, a table of class numbers [5] containing $h(p)$ for the first 5000 primes $p \equiv 1 \pmod 4$, $(5 \leqslant p \leqslant 105269)$ was deposited in the UMT file. In his review of this file, D. Shanks [7] analyzed the data in two ways. Firstly, he followed up Kloss's observation on the proportion of fields with $h(p) = 1$, producing a table showing the number of primes p in each portion of 1000 values, which had a given class number. He noted that the proportion of values p with class numbers $h = 1, 3, 5, 7$ and 9, respectively, was 80%, 10%, 3.6%, 2% and 1.2%; and further that these proportions were remarkably stable in the smaller intervals. This distribution reinforces Gauss's related conjecture [1, Section 304] that the number of fields with one class per genus is a fixed proportion of the population as the determinant goes to infinity. Shanks raises the question on what is the nature of this distribution.

Another more recent tabulation, by Richard B. Lakein [6], is a table of class numbers of the quartic fields $K = F(\pi^{1/2})$ where $F = Q(i)$ for 5000 Gaussian primes $\pi \equiv \pm 1 \pmod 4$. This distribution of class numbers of these fields is strikingly similar to that noted earlier by Shanks [7].

Shanks' second analysis was a table of the exceptionally large class numbers, or more specifically, those primes p, for which $h(p)$ is larger than any preceding class number. This list contained 12 primes ranging from $p = 229$ $(h = 3)$ to $p = 90001$ $(h = 87)$. We can note that these values are bounded above by $\sqrt{p}$, with the ratio $h/\sqrt{p}$ ranging from 0.20 to 0.36.

Received September 29, 1974; revised February 24, 1975.
AMS (MOS) subject classifications (1970). Primary 12A25, 12A50.
Key words and phrases. Ideal classes, continued-fraction expansion, Lagrange algorithm, ideal genera, Dirichlet series, fundamental unit.

TABLE I

g = 1

$e=10^{-4}d$	total	f = 1	3	5	7	9	11	13	15	17	19	21	23	25	27	29	31	33	35	37	39	41	43	45	47	49	+
$0<e<1$	2176	1827	212	76	34	13	5	4	2	1		1			1												
$1<e<2$	1893	1540	179	85	34	19	12	10	7	3	2	1											1				
$2<e<3$	1819	1467	204	56	38	21	13	4	8	1		1	1	2	1				1	1							
$3<e<4$	1762	1453	173	60	23	13	5	9	10	3	5	2	2		3									1			
$4<e<5$	1723	1355	183	65	39	36	14	11	6	4	2	1	1	1		2			1		1						57 (f = 57 occurred once)
$5<e<6$	1687	1314	191	70	32	20	14	8	12	9	3	2		3	2	1		1		2	1	1					63
$6<e<7$	1693	1347	159	65	38	20	16	8	6	6	2	10	4	4	2	1		2						1			51,53
$7<e<8$	1653	1307	184	60	31	25	13	10	5	3	4	3	1			3		1		1					1	1	
$8<e<9$	1645	1307	176	65	30	21	12	11	3	6	2	4		3	1	2		1			1						
$9<e<10$	1632	1317	166	53	25	17	12	9	9		8	3	1	3	1	2	1	1	1	1				1			87
$0<e<10$	17683	14234	1827	655	324	205	116	84	68	36	28	28	10	16	11	11	1	6	3	5	3	1	1	3	1	1	5 (5 occurrences with f > 49)

g = 2

	total	f = 1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	+
$0<e<1$	2847	2016	385	215	84	74	27	20	9	11	2	3	1														
$1<e<2$	2795	1884	370	229	87	76	48	43	13	19	9	4	4	3	1	3		1		1							
$2<e<3$	2763	1833	342	223	96	102	47	40	19	22	8	5	5	5	3	5	1	4	1	1							29
$3<e<4$	2747	1785	343	229	93	91	59	42	30	19	16	8	6	10	4	5	2	1	1	2							37
$4<e<5$	2743	1794	354	212	87	83	37	40	30	33	19	13	9	9	4	5	2	3	1	2	2	1		1			27,36
$5<e<6$	2759	1789	362	228	92	77	51	43	21	27	18	12	10	2	3	6	3	4	1	2		2	1	2		1	34,35
$6<e<7$	2683	1708	359	237	94	88	42	30	21	27	10	13	8	8	8	2	5	4	1	6	4	3	2			2	31
$7<e<8$	2720	1763	343	247	86	90	46	26	21	22	13	9	12	6	8	9	2	4	1	1	1	2	2	1	1		26,29,30,35
$8<e<9$	2690	1703	354	237	87	67	54	42	19	24	23	17	10	10	4	12	6	3	5	3		3	1		1	2	28,30,33
$9<e<10$	2686	1694	358	234	86	80	57	41	22	32	12	11	10	7	4	9	8	1	4	3	1	1	1	2			27,28,29,30,30,35,38,47
$0<e<10$	27433	17969	3570	2291	892	828	468	367	205	236	130	95	75	60	39	56	29	25	15	21	8	12	7	6	2	5	22

TABLE I (*continued*)

g = 4

	total	f = 1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
0<e<1	984	739	145	70	19	9	1	1																	
1<e<2	1242	837	191	114	50	23	13	7	1	2	2	2													
2<e<3	1305	857	202	109	58	41	20	11	3	2			1	1											
3<e<4	1350	869	202	120	65	33	24	16	7	6	1	4		2	1										
4<e<5	1392	891	201	117	60	43	30	21	12	9	1	2	2	3											
5<e<6	1362	839	221	124	73	39	28	15	9	7	3	2	1												1
6<e<7	1455	914	227	120	63	52	25	20	8	14	3	3	2	1	2			1							
7<e<8	1415	889	197	111	79	44	27	23	9	13	8	5	2		6		1					1			
8<e<9	1470	909	233	119	63	52	31	16	12	13	6	7	1	1	1	2	2	1				1			
9<e<10	1470	946	196	111	69	33	44	23	20	8	5	6		2		2	1	2		1	1				
0<e<10	13445	8690	2015	1115	599	369	243	153	81	74	29	31	9	10	10	4	4	4		1	1	2			1

g = 8

	total	f = 1	2	3	4	5	6	7	8	9	10	11
0 < e < 1	75	63	12									
1 < e < 2	145	111	22	9	2	1						
2 < e < 3	194	140	42	8	3	1						
3 < e < 4	204	132	40	23	5	3	1					
4 < e < 5	225	144	43	23	10	3	1	1				
5 < e < 6	257	169	47	21	11	5	3					1
6 < e < 7	244	167	36	21	10	8	1			1		
7 < e < 8	276	184	43	25	16	5	2					1
8 < e < 9	267	180	50	25	5	1	3	1	2			
9 < e < 10	280	165	59	26	12	12	2	2	2			
0 < e < 10	2167	1455	394	181	74	39	13	4	4	1		2

g = 16

	total	f = 1	2	3
0 < e < 1	0			
1 < e < 2	2	2		
2 < e < 3	1	1		
3 < e < 4	5	4	1	
4 < e < 5	8	7	1	
5 < e < 6	7	4	3	
6 < e < 7	8	6	2	
7 < e < 8	7	6	1	
8 < e < 9	14	10	2	2
9 < e < 10	13	9	2	2
0 < e < 10	65	49	12	4

Recently the opportunity arose during the testing and early operation of a Burroughs B6700 computer to run a job which was "computer bound" for long intervals with little input output and a small usage of core.

A program was written based on a class number algorithm which is similar to Ince's procedure of counting periods. Briefly, this algorithm was based on an adaption of Lagrange's algorithm for calculating the continued fraction coefficients of a quadratic surd as described in [2]. If we apply this algorithm to the quadratic surd β/b, where β is a primitive algebraic integer of the real quadratic field of discriminant D, and $b \,\|\, N(\beta)$, $b < \frac{1}{2}D^{1/2}$, then we find that the sequence Q_n in the algorithm includes the norms of all the primitive ideals in the same class as $B = (b, \beta)$, with norm $< \frac{1}{2}D^{1/2}$. Hence, by using the Minkowski bound, we can calculate the class number. The program calculated the class number, $h(d)$, the number of ideal genera, $g(d)$ and the number of ideal classes of the principal genus, $f(d)$ for each field $Q(\sqrt{d})$ with squarefree determinant d, $1000 < d < 100000$. A table was produced, in 99 sections, each section giving $d, f(d), g(d), h(d)$ for each squarefree d in an interval of 1000 integers, followed by a count of the number fields with each combination $(g(d), f(d))$ that occurred.

2. Checking the Table. Several partial checks were made. The most important was that $g(d)$, $h(d)$ were calculated independently, and a check was made to ensure that their ratio $f = h/g$ was integral. This check is, of course, of no value for $g = 1$. Secondly, the values of $h(d)$ for the first 1227 values $1 < d < 2025$ were computed and compared with the corresponding class numbers in Ince's table [3]. Finally, the 16 extreme values of $h(p)$ extracted by Shanks in [7] were checked against the table.

3. Distribution. The distribution of the class numbers of the fields is given in Table I. For squarefree integers d in a given interval, let N_g be the number of fields $Q(\sqrt{d})$ with g genera, and let N_{fg} be the number of these which have f ideal classes per genus. The entries in Table I are the values N_g and N_{fg} for values of d in intervals of 10000 integers. If we consider the proportions N_{fg}/N_g we find for small values of h, these proportions appear to be independent of d, although in the tail of the distribution with h relatively large, the proportion increases with d. A plot of the value N_{fg}/N_g against f on log-log paper suggested that for small values of h, $N_{fg} \doteqdot N_g/f^2$. This would mean that for $g = 1$, $N_{fg}/N_g \doteqdot 8\pi^{-2}f^{-2}$ (odd values of f only), and for $g > 1$, $N_{fg}/N_g \doteqdot 6\pi^{-2}f^{-2}$.

In Table II we compare the proportions N_{1f}/N_1 for the total population, with

TABLE II

$f =$	1	3	5	7	9	11	13	15	17	19	+
$8\pi^{-2}f^{-2}$	81.1	9.0	3.2	1.7	1.0	0.67	0.48	0.36	0.28	0.22	2.0
N_{1f}/N_1	80.5	10.3	3.7	1.8	1.2	0.65	0.48	0.38	0.20	0.16	0.60
$N_f^{(1)}/5000$	79.7	10.4	3.6	2.0	1.3	0.58	0.56	0.40	0.22	0.22	0.90
$N_f^{(2)}/5000$	79.9	10.5	4.0	1.7	1.1	0.56	0.60	0.42	0.38	0.12	0.76

the values of $6\pi^{-2}f^{-2}$. For comparison we also include $N_f^{(1)}/5000$, and $N_f^{(2)}/5000$, where the $N_f^{(1)}$ are the corresponding values obtained from Kloss's table [7] and the $N_f^{(2)}$ from Lakein's table [6]. The three distributions agree closely although they are from dissimilar populations. Each value is expressed as a percentage.

In Table III we compare the proportions N_{gf}/N_f for $g = 2, 4, 8$ for the total population, with the values of $6\pi^{-2}f^{-2}$ each proportion being given as a percentage.

TABLE III

f =	1	2	3	4	5	6	7	8	9	10	+
g = 2	66.5	13.0	8.4	3.2	3.0	1.7	1.3	0.75	0.86	0.47	1.7
4	64.6	15.0	8.3	4.5	2.8	1.8	1.1	0.60	0.55	0.23	0.57
8	67.1	18.2	8.4	3.4	1.8	0.60	0.18	0.18	0.05	0.00	0.09
$6f^{-2}\pi^{-2}$	60.8	15.2	6.8	3.8	2.4	1.7	1.2	0.95	0.75	0.61	5.8

It can be seen that when h is small ($h < 25$), N_{fgf}/N_g is independent of g for $g = 2, 4$ and 8. Again, the behavior of the tail, for large values of h differs markedly from $6f^{-2}\pi^{-2}$, indeed if $N_{gf} = 6f^{-2}\pi^{-2}N_g$, we would find the average value of f over all fields in the interval with g genera was

$$\left(\sum_{f=1}^{\infty} f \cdot N_{gf}\right)\Big/N_g = 6(8)\pi^{-2}\sum_{f=1}^{\infty} f^{-1} = \infty,$$

as opposed to the measured average values, 1.96, 2.09, 1.89, 1.59 and 1.31 for the values $g = 1, 2, 4, 8$ and 16.

If we compute the values of $\pi^{-1}f^{-1}(6(8)N_g/N_{fg})^{1/2}$ for $5 < e < 10$, we find that for values of $h \leqslant 25$, this value ranges from 1.46 ($N_{1,23} = 6$) to 0.87 ($N_{8,3} = 118$), with an average value of 0.94. Hence, we conjecture, for a range of values of d in the neighborhood of the integer a,

TABLE IV

f	d	f	d	f	d	f	d	f	d
g=1		43	14401	13	13321	36	45511	19	99295
1	2*	45	32401	14	11794	37	38026*	20	94546
3	79*	47	78401	15	11321	38	93619	21	77779
5	401	49	70969	16	25282	47	99226	24	50626*
7	577	51	69697	17	19882	g=4		g=8	
9	1129	53	69694	18	25279	1	130	1	1155
11	1297	57	41617	19	19834	2	399	2	4354
13	4759	63	57601	20	41599	3	730*	3	10455
15	9871	87	90001	21	47959	4	3026	4	16555
17	7054	g=2		22	59203	5	3970	5	19210
19	15409	1	10*	23	49321	6	9790	6	33490
21	7057	2	82*	24	79522	7	5626*	7	48399
23	23593	3	235	25	54769	8	16555	8	81130
25	24859	4	226*	26	77842	9	18226	9	65026
27	8761*	5	1111	27	49834	10	16899	11	56170
29	49281	6	1522	28	84679	11	11026*	g=16	
31	97753	7	1534*	29	27226	12	21610*	1	15015
33	55339	8	2305*	30	78745	13	23410*	2	39270
35	25601	9	4954	31	68179	14	39999	3	81510
37	24337	10	3601*	33	87271	15	88231		
39	41614	11	4762*	34	53362	16	71290		
41	55966	12	9634	35	56011	17	63505		

CONJECTURE A. For a large, h small, N_{gf}/N_g is independent of a.

CONJECTURE B. For a large, h small, $N_{gf} \doteqdot N_{g1}/f^2$.

4. Extreme Values. In Table IV we list the least determinant d $(d < 100000)$ which generates the quadratic number field $Q(\sqrt{d})$ with g genera and f ideal classes per genus.

In Table IV the values of d with an asterisk are those values d for which $h(d)$ is larger than any preceding class number.

h can be obtained analytically from the Dirichlet series $L(1, \chi)$ where χ is the character of the field $Q(\sqrt{d})$ using the formula, $h = L(1, \chi)\sqrt{D}/2 \ln \epsilon$, D being the discriminant of $Q(\sqrt{d})$, ϵ the fundamental unit. In [7] Shanks notes $L(1, \chi) = O(\ln \sqrt{d})$. Also as $\epsilon = (x + y\sqrt{D})/2$, with $x, y \geqslant 1, x^2 - Dy^2 = \pm 4 \Rightarrow x > \sqrt{D} - 2$, $\epsilon > \sqrt{D} - 1$, and hence $h = O(\sqrt{d})$. In Table V we give those values of Table IV for which $h/\sqrt{d} > 0.4$.

TABLE V

d	2	10	226	82	730	50626	11026	399
h	1	2	8	4	12	96	44	8
$h/\sqrt{d}$	0.71	0.63	0.53	0.44	0.44	0.43	0.42	0.40.

Mathematics Department
Massey University
Palmerston North, New Zealand

1. C. F. GAUSS, *Disquisitiones Arithmeticae,* Fleischer, Leipzig, 1870; English transl., Yale Univ. Press, New Haven, Conn., 1966. MR 33 #5545.

2. M. D. HENDY, "Applications of a continued fraction algorithm to some class number problems," *Math. Comp.,* v. 28, 1974, pp. 267–277. MR 48 #8440.

3. E. L. INCE, *Cycles of Reduced Ideals in Quadratic Fields,* Mathematical Tables, vol. IV, British Association for the Advancement of Science, London, 1934.

4. K. E. KLOSS, "Some number-theoretic calculations," *J. Res. Nat. Bur. Standards Sect. B,* v. 69B, 1965, pp. 335–336. MR 32 #7473.

5. K. E. KLOSS, M. NEWMAN & E. ORDMAN, *Class Number of Primes of the Form 4n + 1*, National Bureau of Standards, 1965. (Deposited in the UMT file.)

6. R. B. LAKEIN, "Computation of the ideal class group of certain complex quartic fields," *Math. Comp.,* v. 28, 1974, pp. 839–846.

7. D. SHANKS, Review of UMT file [5], *Math. Comp.,* v. 23, 1969, pp. 213–214.